mon factor $R - r$ to obtain $3\rho^3 + 5\rho^2 - 2 = 0$, the roots of this equation being

$$-1, \qquad -(\sqrt{7}+1)/3, \qquad (\sqrt{7}-1)/3.$$

Substituting the positive root in the following modified form of equation (4)

$$R^6 = \frac{9(1+2\rho)v^2}{2(1-\rho^3)(1+\rho+\rho^2)^2\pi^2},$$

we get

$$R = \frac{(88+13\sqrt{7})^{1/6}v^{1/3}}{2^{1/6}\cdot 7^{1/4}\pi^{1/3}} \doteq 0.833293v^{1/3},$$

and $r = \frac{1}{3}(\sqrt{7}-1)R \doteq 0.457131v^{1/3}$.

The values of h and s may now be obtained from formulas (2) and (6) respectively. They are $h = [2(\sqrt{7}-2)v/\pi]^{1/3} \doteq 0.743559v^{1/3}$, $s = R$.

The last relation is the most significant geometrically. Let θ denote the acute angle between the axis and the slant height of the frustum. Then

$$\theta = \sin^{-1}[(R-r)/s] = \sin^{-1}[(4-\sqrt{7})/3] \doteq 26° \, 50' \, 4.5''.$$

From the practical point of view, the problem amounts to asking for the smallest amount of sheet metal out of which a cup or deep cake pan of given capacity can be made.

2870 [1921, 36]. Proposed by WARREN WEAVER, University of Wisconsin.

A pendulum bob of mass m is attached to one end of a weightless and inextensible string of length l and swings as a conical pendulum with an angular velocity ω_1 about a vertical line through a fixed point to which the other end of the string is attached. If the angular velocity is increased to ω_2, the height through which the bob rises is independent of the length l. Consider, then, a very long and a very short pendulum. Suppose they are each swung first with an angular velocity ω_1 and then with a larger angular velocity, ω_2, the difference between these two values being great enough so that the longer pendulum rises through a height greater than the length of the shorter pendulum. According to the above result, the shorter one should rise through this same height, which is obviously impossible. Explain this apparent paradox.

SOLUTION BY I. MAIZLISH, University of Minnesota.

Let l be the length of the string, α_1 the angle which the equilibrium position of the pendulum makes with the vertical when the angular velocity is ω_1, and α_2 the corresponding angle when the angular velocity is ω_2. Assume $\omega_2 > \omega_1$, that the pendulum is free to take a vertical position when not revolving, and that ω_1 and ω_2 are finite. It is readily seen that for equilibrium the following equation must hold:

$$\tan\alpha_1 = (\omega_1^2 l \sin\alpha_1)/g, \tag{1}$$

and if $\alpha_1 > 0$ we must have

$$\omega_1 > \sqrt{(g/l)}. \tag{2}$$

The height of the bob from the horizontal plane passing through it when the pendulum is in its vertical position is $h_1 = l(1-\cos\alpha_1) = l - (g/\omega_1^2)$, and if the angular velocity is increased from ω_1 to ω_2, α being increased to α_2, the height through which the bob rises will be

$$h_2 - h_1 = (g/\omega_1^2) - (g/\omega_2^2), \tag{3}$$

as long as $\omega_1 \geqq \sqrt{(g/l)}$.

Let l' be the length of the short pendulum. If ω_1 and ω_2 are chosen so that $(g/\omega_1^2) - (g/\omega_2^2) > l'$ we shall have $\omega_1 < \sqrt{(g/l')}$ and we cannot apply (3) to this pendulum. If $\omega_2 > \sqrt{(g/l')}$ and the bob of the short pendulum rises at all, the height to which it rises will be $l' - g/\omega_2^2$, but if $\omega_2 \leqq \sqrt{(g/l')}$, it will not rise at all.

Also solved by H. L. OLSON, F. L. WILMER and the PROPOSER.

2872 [1921, 36]. Proposed by W. D. LAMBERT, U. S. Coast and Geodetic Survey.

The rectangular coördinates of a point P at the time t are given by the equations

$$x = k\cos\gamma\cos(nt-\alpha), \qquad y = k\sin\gamma\cos(nt-\beta),$$

where k, γ, n, α, and β are constants and γ is taken in the first quadrant. An auxiliary angle δ, in the first quadrant, is defined by the equation

$$\sin \delta = \sin 2\gamma \,|\sin (\alpha - \beta)|.$$

(1) Show that P describes an ellipse the lengths of whose semi-axes are $k \cos \frac{1}{2}\delta$ and $k \sin \frac{1}{2}\delta$ and the inclinations of whose axes to the x-axis are given by

$$\tan 2\theta = \tan 2\gamma \cos (\alpha - \beta).$$

(2) Show that the time T when P is at the end of an axis is given by

$$\tan (2nT - \alpha - \beta) = \cos 2\gamma \tan (\alpha - \beta).$$

(3) Obtain criteria for distinguishing between the major and minor axes and for the direction of rotation of P and show that the quantities γ, $\alpha - \beta$, δ, θ and $nT - \alpha - \beta$ are simply related as the parts of a right spherical triangle, thus providing a partial check on the computation.

Solution by J. B. Reynolds, Lehigh University.

1. Eliminating t from the given parametric equations, we have

$$\alpha + \cos^{-1} \frac{x \sec \gamma}{k} = \beta + \cos^{-1} \frac{y \csc \gamma}{k},$$

or

$$\cos (\alpha - \beta) = \frac{xy \sec \gamma \csc \gamma}{k^2} + \frac{1}{k^2} \sqrt{k^2 - x^2 \sec^2 \gamma}\, \sqrt{k^2 - y^2 \csc^2 \gamma};$$

whence

$$x^2 \tan^2 \gamma + y^2 - 2xy \cos (\alpha - \beta) \tan \gamma - k^2 \sin^2 (\alpha - \beta) \sin^2 \gamma = 0 \tag{1}$$

is the rectangular equation of the curve, and is that of an ellipse with center at the origin. To ascertain the length s of the axes then, we need find only the maximum and minimum values of $r = \sqrt{x^2 + y^2}$. Now,

$$r^2 = x^2 + y^2 = k^2[\cos^2 \gamma \cos^2 (nt - \alpha) + \sin^2 \gamma \cos^2 (nt - \beta)]. \tag{2}$$

Setting the $d(r^2)/dt = 0$, we get

$$\tan 2nt = \frac{\cos^2 \gamma \sin 2\alpha + \sin^2 \gamma \sin 2\beta}{\cos^2 \gamma \cos 2\alpha + \sin^2 \gamma \cos 2\beta} = \frac{S}{C}, \quad \text{say;} \tag{3}$$

and we find

$$\sqrt{S^2 + C^2} = \sqrt{1 - \sin^2 2\gamma \sin^2 (\alpha - \beta)} = \cos \delta \text{ by the data given;}$$

whence

$$\sin 2nt = \pm \frac{S}{\cos \delta} \qquad \text{and} \qquad \cos 2nt = \pm \frac{C}{\cos \delta}.$$

Putting these values in (2), we get for maximum and minimum values of r^2,

$$\frac{k^2}{2}\left[1 \pm \left(\frac{S^2 + C^2}{\cos \delta}\right)\right] = \frac{k^2}{2}(1 \pm \cos \delta),$$

so that the major and minor semi-axes are $k \cos \delta/2$ and $k \sin \delta/2$. The angle made by these axes with the x-axis comes immediately from (1) by writing the condition for elimination of the term in xy or[1]

$$\tan 2\theta = \frac{-2 \cos (\alpha - \beta) \tan \gamma}{\tan^2 \gamma - 1} = \tan 2\gamma \cos (\alpha - \beta).$$

2. When the ends of the axes are reached, r is a maximum or minimum. In getting $d(r^2)/dt = 0$, we have

$$\cos^2 \gamma \sin (2nt - 2\alpha) + \sin^2 \gamma \sin (2nt - 2\beta) = 0.$$

[1] Similarly the expression for the semi-axes could be obtained by use of the invariants of rotation, and then the equation for T could be derived by algebraic processes. On the other hand, the equation for θ ($\tan \theta$ being y/x) could be deduced from the equation $d(r^2)/dt = 0$ without any reference to the theory of rotation of axes.—Editors.

Or, putting $\frac{1}{2}(1 + \cos 2\gamma)$ and $\frac{1}{2}(1 - \cos 2\gamma)$ for $\cos^2 \gamma$ and $\sin^2 \gamma$,

$$\sin (2nt - \alpha - \beta) \cos (\alpha - \beta) - \cos 2\gamma \cos (2nt - \alpha - \beta) \sin (\alpha - \beta) = 0;$$

and then replacing t by the particular value T at this time,

$$\tan (2nT - \alpha - \beta) = \cos 2\gamma \tan (\alpha - \beta).$$

3. The major semi-axis is $k \cos \delta/2$, δ being by hypothesis an angle in the first quadrant. Differentiating

$$\frac{y}{x} = \tan \gamma \frac{\cos (nt - \beta)}{\cos (nt - \alpha)},$$

we have for the angular velocity of P

$$\frac{n \tan \gamma \sin (\beta - \alpha)}{\cos^2 (nt - \alpha)},$$

positive or negative with $\sin (\beta - \alpha)$.

We can place 2γ on the hypotenuse, 2θ and δ on the legs of a right spherical triangle, $90° - (2nT - \alpha - \beta)$ opposite 2θ and $\alpha - \beta$ opposite δ on the angles; for the three equations

$$\sin \delta = \sin 2\gamma \sin (\alpha - \beta), \qquad \cos (\alpha - \beta) = \tan 2\theta \cot 2\gamma,$$
$$\cos 2\gamma = \cot (\alpha - \beta) \tan (2nT - \alpha - \beta)$$

will then conform to Napier's rules.[1]

2878 [1921, 89]. Proposed by R. S. HOAR, Fort Banks, Mass.

Consider the integers $0, 1, 2, 3 \cdots n - 1, n$. Consider all possible permutations of combinations of these taken r at a time, allowing any integer to occur more than once. Select from these permutations all groups the sum of whose integers is n. Form the reciprocal of the product of the factorials of the r integers of each of these selected groups. Then the sum of all of these reciprocals will equal $r^n/n!$. Prove that this must be so.

Example. $n = 2$, $r = 3$.

$$\frac{1}{2!0!0!} + \frac{1}{0!0!2!} + \frac{1}{0!2!0!} + \frac{1}{1!0!1!} + \frac{1}{1!1!0!} + \frac{1}{0!1!1!} = \frac{3^2}{2!}, \quad \text{if} \quad 0! = 1.$$

SOLUTION BY H. L. OLSON, University of Michigan, and PHILIP FRANKLIN, Princeton University.

By the multinomial theorem, we have

$$(x_1 + x_2 + \cdots + x_r)^n = \Sigma \frac{n!}{i_1!\, i_2! \cdots i_r!} x_1^{i_1} x_2^{i_2} \cdots x_r^{i_r},$$

where $i_1 + i_2 + \cdots + i_r = n$ and $i_1, i_2, \cdots, i_r \geqq 0$. If we set each x_i equal to unity and divide the two members of the equation by $n!$, we have the theorem stated.

2880 [1921, 89]. Proposed by SIDNEY DORB, Detroit, Mich.

Solve the simultaneous equations $xy = 2$, and

$$\left(3 - \frac{6y}{x - y}\right)^2 + \left(3 - \frac{6y}{x + y}\right)^2 = 82.$$

SOLUTION BY A. A. BENNETT, University of Texas.

It is assumed for various reasons that the second equation was intended in the form

$$\left(3 + \frac{6y}{x - y}\right)^2 + \left(3 - \frac{6y}{x + y}\right)^2 = 82,$$

[1] This supposes $\sin (\alpha - \beta)$ positive, and it would not essentially restrict the problem to assume that α and β are angles in the first quadrant, $\alpha > \beta$.